A note on "Sur le noyau de l'opérateur de courbure d'une variété finslérienne, C. R. Acad. Sci. Paris, sér. A, t. 272 (1971), 807-810"*

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Abstract. In this note, adopting the pullback formalism of global Finsler geometry, we show by a counterexample that the kernel Ker_R of the h-curvature R of Cartan connection and the associated nullity distribution $\mathcal{N}_R$ do not coincide, contrary to Akbar-Zadeh's result [1]. We also give sufficient conditions for Ker_R and $\mathcal{N}_R$ to coincide.

Keywords: Cartan connection, h-curvature tensor, Nullity distribution, Kernel distribution.

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1. Introduction and notations

Nullity distribution in Finsler geometry has been investigated in [1] (adopting the pullback formalism) and [5] (adopting the Klein-Grifone formalism). In 1971, Akbar-Zadeh [1] proved that the kernel Ker_R of the h-curvature operator R of Cartan connection coincides with the nullity distribution $\mathcal{N}_R$ of that operator. This result was reappeared again in [2] and was used to prove that the nullity foliation is auto-parallel. Moreover, Bidabad and Refie-Rad [3] generalized this result to the case of k-nullity distribution following the same pattern of proof as Akbar-Zadeh's.

In this note, we show by a counterexample that Ker_R and $\mathcal{N}_R$ do not coincide, contrary to Akbar-Zadeh's result. In addition, we find sufficient conditions for Ker_R and $\mathcal{N}_R$ to coincide.

In what follows, we denote by $\pi : \mathcal{T}M \longrightarrow M$ the subbundle of nonzero vectors tangent to M, $\pi_* : T(\mathcal{T}M) \longrightarrow TM$ the linear tangent map of π and $V_z(TM) = (\text{Ker}\,\pi_*)_z$ the vertical space at $z \in \mathcal{T}M$. Let $\mathfrak{F}(TM)$ be the algebra of C^∞ functions on TM and $\mathfrak{X}(\pi(M))$ the $\mathfrak{F}(TM)$-module of differentiable sections of the pullback bundle $\pi^{-1}(TM)$. The elements of $\mathfrak{X}(\pi(M))$

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will be called π-vector fields and denoted by barred letters $\overline{X}$. The fundamental π-vector field is the π-vector field $\overline{\eta}$ defined by $\overline{\eta}(z) = (z, z)$ for all $z \in \mathcal{T}M$.

Let D be a linear connection on the pullback bundle $\pi^{-1}(TM)$. Let K be the map defined by $K : T(\mathcal{T}M) \longrightarrow \pi^{-1}(TM) : X \longmapsto D_X\overline{\eta}$. The vector space $H_z(TM) := \{X \in T_z(\mathcal{T}M) : K(X) = 0\}$ is the horizontal space to M at z. The restriction of π_* on $H_z(TM)$, denoted again π_*, defines an isomorphism between $H_z(TM)$ and $T_{\pi z}M$. The connection D is said to be regular if $T_z(\mathcal{T}M) = V_z(TM) \oplus H_z(TM)\ \forall\, z \in \mathcal{T}M$. In this case K defines an isomorphism between $V_z(TM)$ and $T_{\pi z}M$.

If M is endowed with a regular connection, then the preceding decomposition permits to write uniquely a vector $X \in T_z(\mathcal{T}M)$ in the form $X = hX + vX$, where $hX \in H_z(TM)$ and $vX \in V_z(TM)$. The ((h)hv-) torsion tensor of D, denoted by T, is defined by $T(\overline{X}, \overline{Y}) = \mathbf{T}(vX, hY)$, for all $\overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M))$, where $\mathbf{T}(X,Y) = D_X\overline{Y} - D_Y\overline{X} - \pi_*[X,Y]$ is the (classical) torsion associated with D and $\overline{X} = \pi_* X$ (the fibers of the pullback bundle are isomorphic to the fibers of the tangent bundle). The h-curvature tensor of D, denoted by R, is defined by $R(\overline{X}, \overline{Y})\overline{Z} = \mathbf{K}(hX, hY)\overline{Z}$, where $\mathbf{K}(X,Y)\overline{Z} = D_X D_Y \overline{Z} - D_Y D_X \overline{Z} - D_{[X,Y]}\overline{Z}$ is the (classical) curvature associated with D. The contracted curvature $\widehat{R}$ is defined by $\widehat{R}(\overline{X}, \overline{Y}) = R(\overline{X}, \overline{Y})\overline{\eta}$.

2. Kernel and nullity distributions: Counterexample

Let (M, F) be a Finsler manifold. Let ∇ be the Cartan connection associated with (M, F). It is well known that ∇ is the unique metrical regular connection on $\pi^{-1}(TM)$ such that $g(T(\overline{X}, \overline{Y}), \overline{Z}) = g(T(\overline{X}, \overline{Z}), \overline{Y})$ [2], [6]. Note that the bracket $[X, Y]$ is horizontal if and only if $\widehat{R}(\overline{X}, \overline{Y}) = 0$, where $\widehat{R}$ is the contracted curvature of the h-curvature tensor of ∇.

Lemma 2.1. *[2] Let $\boldsymbol{T}$ and $\boldsymbol{K}$ be the (classical) torsion and curvature tensors of ∇ respectively. We have:*

$$\mathfrak{S}_{X,Y,Z}\{\boldsymbol{K}(X,Y)\overline{Z} - \nabla_Z \boldsymbol{T}(X,Y) - \boldsymbol{T}(X,[Y,Z])\} = 0,$$

where the symbol $\mathfrak{S}_{X,Y,Z}$ denotes cyclic sum over $X, Y, Z \in \mathfrak{X}(TM)$.

Let us now define the concepts of nullity and kernel spaces associated with the curvature $\mathbf{K}$ of ∇, following Akbar-Zadeh's definitions [1].

Definition 2.2. *The subspace $\mathcal{N}_{\mathbf{K}}(z)$ of $H_z(TM)$ at a point $z \in TM$ is defined by*

$$\mathcal{N}_{\mathbf{K}}(z) := \{X \in H_z(TM) : \mathbf{K}(X,Y) = 0,\ \forall Y \in H_z(TM)\}.$$

The dimension of $\mathcal{N}_{\mathbf{K}}(z)$ is denoted by $\mu_{\mathbf{K}}(z)$.

The subspace $\mathcal{N}_{\mathbf{K}}(x) := \pi_*(\mathcal{N}_{\mathbf{K}}(z)) \subset T_xM$, $x = \pi z$, is linearly isomorphic to $\mathcal{N}_{\mathbf{K}}(z)$. This subspace is called the nullity space of the curvature operator $\boldsymbol{K}$ at the point $x \in M$

Definition 2.3. *The kernel of $\mathbf{K}$ at the point $x = \pi z$ is defined by*

$$\mathrm{Ker}_{\mathbf{K}}(x) := \{\overline{X} \in \{z\} \times T_xM \simeq T_xM : \mathbf{K}(Y,Z)\overline{X} = 0, \forall Y, Z \in H_z(TM)\}.$$

Since $\mathcal{N}_{\mathbf{K}}$ and $\mathrm{Ker}_{\mathbf{K}}$ are both defined on the horizontal space, we can replace the classical curvature $\mathbf{K}$ by the h-curvature tensor R of Cartan connection. Akbar-Zadeh [1] proved that the nullity space $\mathcal{N}_{\mathbf{K}}(x)$ and the kernel space $\mathrm{Ker}_{\mathbf{K}}(x)$ coincide for each point $x \in M$ at which they are defined. We show by a counterexample that the above mentioned spaces do not coincide.

Theorem 2.4. *The nullity space* $\mathcal{N}_R(x)$ *and the kernel space* $\mathrm{Ker}_R(x)$ *do not coincide.*

Let $M=\mathbb{R}^3$, $U=\{(x_1,x_2,x_3;y_1,y_2,y_3)\in\mathbb{R}^3\times\mathbb{R}^3 : x_3y_1>0,\ y_2^2+y_3^2\neq 0\}\subset TM$. Let F be the Finsler function defined on U by

$$F:=\sqrt{x_3y_1\sqrt{{y_2}^2+{y_3}^2}}.$$

Using MAPLE program, we can perform the following computations. We write only the coefficients Γ^i_j of Barthel connection and the components R^h_{ijk} of the h-curvature tensor R.
The non-vanishing coefficients of Barthel connection Γ^i_j are:

$$\Gamma^2_2=\frac{y_3}{x_3},\qquad \Gamma^2_3=\frac{y_2}{x_3},\qquad \Gamma^3_2=-\frac{y_2}{x_3},\qquad \Gamma^3_3=\frac{y_3}{x_3}.$$

The independent non-vanishing components of the h-curvature R^h_{ijk} of Cartan connection are:

$$R^1_{223}=\frac{y_1y_3}{2x_3^2(y_2^2+y_3^2)},\qquad R^1_{323}=-\frac{y_1y_2}{2x_3^2(y_2^2+y_3^2)},\qquad R^2_{123}=-\frac{y_3}{2x_3^2y_1},$$
$$R^2_{323}=-\frac{1}{2x_3^2},\qquad R^3_{123}=\frac{y_2}{2x_3^2y_1},\qquad R^3_{223}=\frac{1}{2x_3^2}.$$

Now, let $X\in\mathcal{N}_R$, then X can be written in the form $X=X^1h_1+X^2h_2+X^3h_3$, where X^1,X^2,X^3 are the components of the vector X with respect to the basis $\{h_1,h_2,h_3\}$ of the horizontal space; $h_i:=\frac{\partial}{\partial x^i}-\Gamma^m_i\frac{\partial}{\partial y^m}$, $i,m=1,...,3$. The equation $R(\overline{X},\overline{Y})\overline{Z}=0$, $\forall\, Y,Z\in H(TM)$, is written locally in the form $X^jR^h_{ijk}=0$. This is equivalent to the system of equations $X^2=0$, $X^3=0$ having the solution $X^1=t\,(t\in\mathbb{R})$, $X^2=X^3=0$. As $\pi_*(h_i)=\frac{\partial}{\partial x^i}$, we have

$$\mathcal{N}_R(x)=\left\{t\frac{\partial}{\partial x^1}\,|\,t\in\mathbb{R}\right\}. \tag{2.1}$$

On the other hand, let $Z\in\mathrm{Ker}_R$. The equation $R(\overline{X},\overline{Y})\overline{Z}=0$, $\forall\, X,Y\in H(TM)$, is written locally in the form $Z^iR^h_{ijk}=0$. This is equivalent to the system:

$$y_3Z^2-y_2Z^3=0,\qquad y_3Z^1+y_1Z^3=0,\qquad y_2Z^1+y_1Z^2=0.$$

This system has the solution $Z^1=t$, $Z^2=-\frac{y_2}{y_1}t$ and $Z^3=-\frac{y_3}{y_1}t$, $(t\in\mathbb{R})$. Thus,

$$\mathrm{Ker}_R(x)=\left\{t\Big(\frac{\partial}{\partial x^1}-\frac{y_2}{y_1}\frac{\partial}{\partial x^2}-\frac{y_3}{y_1}\frac{\partial}{\partial x^3}\Big)\,|\,t\in\mathbb{R}\right\}. \tag{2.2}$$

Comparing (2.1) and (2.2), we note that there is no value of t for which $\mathcal{N}_R(x)=\mathrm{Ker}_R(x)$. Consequently, $\mathcal{N}_R(x)$ and $\mathrm{Ker}_R(x)$ can not coincide. □

According to Akabr-Zadeh's proof, if $X\in\mathcal{N}_R$, then, by Lemma 2.1, we have $R(\overline{Y},\overline{Z})\overline{X}=\mathbf{T}(X,[Y,Z])$. But there is no guarantee for the vanishing of the right-hand side. Even the equation $g(R(\overline{Y},\overline{Z})\pi_*X,\pi_*W)=g(\mathbf{T}(X,[Y,Z]),\pi_*W)$, $W\in H(TM)$, is true only for $X\in\mathcal{N}_R$ and, consequently, we can not use the symmetry or skew-symmetry properties in X and W to conclude that $g(R(\overline{Y},\overline{Z})\overline{X},\overline{W})=0$. This can be assured, again, by the previous example: if we take $X=h_1\in\mathcal{N}_R(z)$ and $Y=h_2, Z=h_3$, then the bracket $[Y,Z]=-\frac{y_3}{x_3^2}\frac{\partial}{\partial y_2}+\frac{y_2}{x_3^2}\frac{\partial}{\partial y_3}$ is vertical and $\mathbf{T}(h_1,[h_2,h_3])=-\frac{1}{2x_3^2y_1}(y_3\bar\partial_2-y_2\bar\partial_3)\neq 0$, where $\bar\partial_i$ is the basis of the fibers of the pullback bundle.

As has been shown above, $\mathcal{N}_R$ and Ker_R do not coincide in general. Nevertheless, we have

Theorem 2.5. *Let (M, F) be a Finsler manifold and R the h-curvatire of Cartan connection. If*

$$\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}} R(\overline{X}, \overline{Y})\overline{Z} = 0, \tag{2.3}$$

then the two distributions $\mathcal{N}_R$ and Ker$_R$ coincide.

Proof. If $X \in \mathcal{N}_R$, then, from (2.3), we have $R(Y, Z)X = 0$ and consequently $X \in \mathrm{Ker}_R$. On the other hand, it follows also from (2.3) that $g(R(\overline{X}, \overline{Y})\overline{Z}, \overline{W}) =: R(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = R(\overline{Z}, \overline{W}, \overline{X}, \overline{Y})$. This proves that if $X \in \mathrm{Ker}_R$, then $X \in \mathcal{N}_R$. □

The following corollary shows that there are nontrivial cases in which (2.3) is verified and consequently the two distributions coincide.

Corollary 2.6. *Let (M, F) be a Finsler manifold and g the associated Finsler metric. If one of the following conditions holds:*

(a) $\widehat{R} = 0$ *(the integrability condition for the horizontal distribution),*

(b) $\widehat{R}(\overline{X}, \overline{Y}) = \lambda F(\ell(\overline{X})\overline{Y} - \ell(\overline{Y})\overline{X})$, *where $\lambda(x, y)$ is a homogenous function of degree 0 in y and $\ell(\overline{X}) := F^{-1}g(\overline{X}, \overline{\eta})$ (the isotropy condition),*

then the two distributions $\mathcal{N}_R$ and Ker$_R$ coincide.

Proof.
(a) We have $\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{R(\overline{X}, \overline{Y})\overline{Z} - T(\overline{X}, \widehat{R}(\overline{Y}, \overline{Z}))\} = 0$ [7]. Then, if $\widehat{R} = 0$, (2.3) holds.
(b) If $\widehat{R}(\overline{X}, \overline{Y}) = \lambda F(\ell(\overline{X})\overline{Y} - \ell(\overline{Y})\overline{X})$, then, by [4], (2.3) is satisfied. □

Remark 2.7. It should be noted that the identity (2.3) is a sufficient condition for the validity of the identity (2.1) of [1].

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